Assignment 8.

Cauchy Integral Formula, Cauchy Inequalities, harmonic functions.

This assignment is due Wednesday, March 16. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Evaluate the integral
$$\frac{1}{2\pi i}\int_L \frac{ze^z}{(z-z_0)^3}dz,$$
where $z_0 \in I(L)$. (*Hint:* Use formula for derivative of Cauchy's integral.)

(2) Suppose that f is differentiable on a disk $B_{10}(i)$ and $|f(z)| \leq 54$ for z on the circle $|z-i| = 3$. Use Cauchy Inequalities to find an upper bound for $|f^{(4)}(i)|$ and for $|f^{(4)}(0)|$.
(*Hint:* For the latter, remember that a strict maximum of absolute value of an analytic function cannot be attained at an interiour point of a set, Theorem 24 in Lecture Notes.)

(3) Suppose f is an entire function such that $|f(z)| \leq M|z|$ for some real constant M. Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$. (*Hint:* Similarly to proof of Liouville's theorem, show that $f'' = 0$.)
Comment. So there are no entire functions "squeezed between" linear and constant ones!

(4) (a) Suppose f is an entire function such that $\operatorname{Re} f$ is bounded. Show that f is constant. (*Hint:* How do you make $|\cdot|$ out of Re?)
(b) Suppose $u(x, y)$ is harmonic on $\mathbb{R}^2$ and bounded. Show that u is constant.

(5) Find a harmonic conjugate for the following functions. (If one does not exist, show that.)
(a) $x^3 + Axy^2$, where A is a real number.
(b) $\sin y \sinh x$.

(6) (a) Suppose u is a harmonic function and v its harmonic conjugate, so $f = u+iv$ is analytic. Show that u^2-v^2 is harmonic by considering f^2.
(b) Generalize the above by proving the following theorem.
Theorem. (Composition of harmonic functions) *Let u, v be conjugate harmonic functions on a domain G. For any function $g = s + it$ analytic on the image of G under $u + iv$, the functions $s(u(x, y), v(x, y))$, $t(u(x, y), v(x, y))$ are harmonic, and the latter is a harmonic conjugate of the former.*
(*Hint:* Do *not* compute $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Instead, argue by complex differentiability.)
(c) Use the above to find a harmonic conjugate for $e^{-2xy}\sin(x^2 - y^2)$.